Octogram versus Pentagram in a (not too important) historical dispute

by

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This short paper deals with a question in the history of Greek mathematics. It appears in this special issue as a token of esteem to Prof. Hayashi, and not, of course, as a contribution to nonlinear mechanics (which happened to be, several years ago, the research subject of one of us). Nevertheless, we have been glad to write it, on the kind insistence of Prof. Rosenberg.

How exactly the Pythagoreans discovered and proved the incommensurability is an open question, reviewed in detail by W.R. KNORR [1]. One of the hypothesis, proposed by K. von FRITZ [2] is that the proof rested upon the Euclidean division algorithm, called *anthyphairesis*, in connection with the figure constituted of an infinite number of nested regular pentagons and star pentagrams (see Fig. 1). The incommensurable segments exhibited by this figure are the side and the diagonal of the regular pentagon, which are to each other in mean and extreme ratio (the so-called golden section). We will not reproduce here the details of the proof of incommensurability in this setting (cf. W.R. Knorr, *op. cit.*)

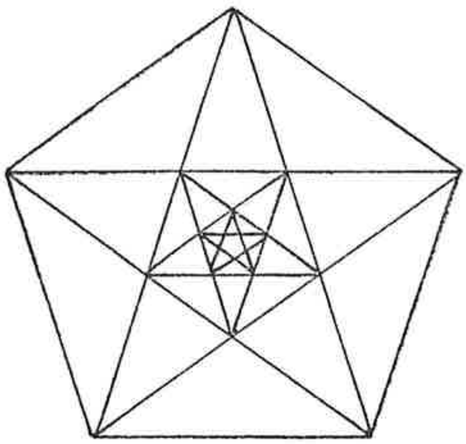

Fig. 1

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Let us now give an analogous proof using Fig. 2, i.e. an infinite number of nested regular octogon~~es~~s and star octograms. We will show that *AB* is incommensurable with *CD*.

In general, the Euclidean algorithm for two segments proceeds as follows. Let us call *a* the larger segment and *b* the smaller. We subtract *b* from *a*, as many times as possible, leaving a remainder *c*, smaller than *b*. Next we subtract *c* from *b*, as many times as possible, etc., etc. If the segments are commensurable, the process terminates, resulting in the greatest common measure. If they are incommensurable, the process continues *ad infinitum*, and the successive remainders become eventually less than any preassigned finite segment.

Consider now Fig. 2 again. Using some obvious geometrical properties, we obtain successively that CD = AE = FB is contained two times in AB, and there remains EF ;

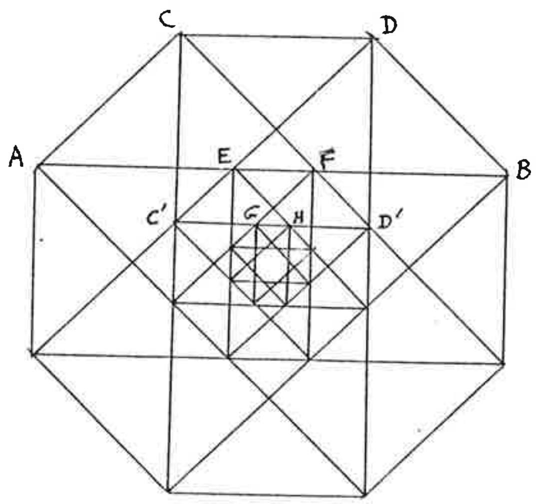

Fig. 2

further CD = C'D', and EF = C'G = HD' is contained two times in C'D', and there remains GH. "The infinite continuation of the process thus becomes evident in this striking way" (W.R. Knorr). Thus AB and CD are incommensurable.

Now W.R. Knorr declares, on the following grounds, that the use of the pentagram by the Pythagoreans to prove the incommensurability is not plausible (we quote in substance) :

1) The division of a line in two segments in mean and extreme ratio requires the general Pythagorean theorem.
2) The construction of the regular pentagon via formally acceptable means is an impressive feat and not one we can assign so easily to the very early stages of geometric inquiry.
3) There is no appearence in Greek litterature of any use of anthyphairesis to *prove* the incommensurability of lines in mean and extreme ratio.

If it is true that the Pythagoreans were familiar with the pentagram, it is highly probable that they also knew other simple polygrams, and the octogram among them. But whereas the pentagram is difficult to construct, the octogram is simple, and the proof outlined above does not use the general case of the Pythagorean theorem, nor any other elaborate geometrical property.

Thus, substituting the octogram for the pentagram in the historical hypothesis of von Fritz seems to remove the arguments of Knorr against its plausibility. This is not a new result, but just a new question. Even if we believe that it makes sense, we will not insist that is is really important.

Bibliography

[1] W. R. KNORR, The Evolution of the Euclidean Elements, D. Reidel, Dordrecht, 1975.

[2] K. von FRITZ, Discovery of Incommensurability by Hippasus of Metapontum, Ann. Math. **46** (1945), 242-264.